Type B Uncertainty Calculator

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Methodology

Typically, a Type B estimate of uncertainty emerges as a cognitive impression based on the recollected experience of a technical expert. In the current paradigm, all that is hoped for is an estimate of uncertainty without accompanying degrees of freedom or other statistics. In the absence of sampled data from which to determine the degrees of freedom associated with an estimate, the degrees of freedom is usually taken to be infinite.

This practice in setting the degrees of freedom for a Type B estimate compromises its use as a statistic in hypothesis testing or in setting confidence limits. We *know* that the estimate is not based on an "infinite" amount of knowledge. In fact, we usually acknowledge that a Type B estimate is made from less knowledge than what typically accompanies a Type A estimate, which is characterized by a finite degrees of freedom. So, the upshot is that the estimates in which we have the least confidence are treated with the most confidence. The problem is exacerbated when attempting to use Welch-Satterthwaite [1, 2] or other means of computing the degrees of freedom for combined Type B and Type A estimates. In these computations, the estimates about which we know the least tend to dominate the end result.

To compensate for the unavailability of rigorous degrees of freedom estimates, an "engineering" solution has been instituted that gives up on the whole idea of determining useful confidence intervals for Type B or mixed Type A/B estimates. In this practice, Type B estimates and mixed estimates are uniformly multiplied by a fixed coverage factor that, hopefully, yields limits that bear some resemblance to confidence limits. In some cases, this practice may produce useful limits, but there is often no way to tell. Unfortunately, all that can truthfully be said about the practice is that, at one point we have an uncertainty estimate and at another point we have k times this estimate. Obviously, we have added nothing to our knowledge or to the utility of the estimate by applying a fixed coverage factor.

What is needed for Type B estimates, is some way to draw from the experience of the estimator both the estimate itself and an accompanying degrees of freedom. It might be pointed out additionally that what is also needed is a means of determining the underlying statistical distribution for the estimate. However, such determinations are rarely made even for estimates obtained from random samples. The usual assumption, which has considerable merit, is to assume an underlying normal distribution. [3-7] This leads to the application of the Student's t distribution in computing confidence intervals. In this monograph we will do likewise with Type B uncertainty estimates.

The approach to be taken is appropriate for the kind of uncertainty-related information that is available to technical experts. This approach begins by formalizing the Type B estimation thought process. This is done by viewing the process as an "experiment" involving independent Bernoulli trials.

Bernoulli Trials and Containment Probability

Suppose we want to find the uncertainty in a variable y from independent Bernoulli trials that each determine (measure) whether the value of y lies within limits $\pm A$. The limits $\pm A$ are referred to herein as **containment limits**.

We define the likelihood function for the *ith* trial of n independent trials in the usual way:

$$L_i = p^{x_i}(1-p)^{1-x_i}$$

where

$$x_i = \begin{cases} 1, \text{ if } y_i \in \pm A \\ 0, \text{ otherwise,} \end{cases}, \; i = 1,2, \cdots, n,$$

and, where p is the probability that y is contained within A. The probability p is referred to as the **containment probability**.

A likelihood function is constructed from the results of the n trials according to

$$L = \prod_{i=1}^{n} L_i = \prod_{i=1}^{n} p^{x_i}(1-p)^{1-x_i} \ ,$$

from whence

$$\ln L = \sum_{i=1}^{n} x_i \ln p + \sum_{i=1}^{n} (1-x_i)\ln(1-p) \ .$$

The containment probability p is estimated by maximizing the likelihood function. This is done by setting the derivative of ln$L$ with respect to p equal to zero

$$\begin{aligned} \frac{\partial}{\partial p} \ln L &= \sum_{i=1}^{n} \frac{x_i}{p} - \sum_{i=1}^{n} \frac{(1-x_i)}{(1-p)} \\ &= \sum_{i=1}^{n} \frac{x_i - p}{p(1-p)} \\ &= 0 \ . \end{aligned} \tag{1}$$

This yields an estimate for p of

$$p = \frac{\sum_{i=1}^{n} x_i}{n} \ , \tag{2}$$

as expected.

The summation in Eq. (2) is the total number of trials measured or observed to lie within $\pm A$. We denote this quantity x:

$$x = \sum_{i=1}^{n} x_i \ ,$$

and write Eq. (2) as

$$p = \frac{x}{n} \ . \tag{3}$$

Estimating Type B Uncertainty

If we assume a distribution for the variable y, then Eq. (3) allows us to estimate the uncertainty in y, based on n observations with outcomes x_1, x_2, ... , x_n. For the sake of discussion, assume that y is normally distributed with zero mean and standard deviation u_y.

Then the uncertainty in y is determined from the containment limits $\pm A$ and the containment probability p according to

$$\begin{aligned} p &= \frac{1}{\sqrt{2\pi} u_y} \int_{-A}^{A} e^{-y^2/2u_y^2} dy \\ &= 2\Phi(A/u_y) - 1, \end{aligned}$$

so that

$$u_y = \frac{A}{\Phi^{-1}\left[(1+p)/2\right]} , \tag{4}$$

where $\Phi(\cdot)$ is the normal distribution function and $\Phi^{-1}(\cdot)$ is the inverse function.[1] Substituting from Eq. (3) yields a "sample" standard deviation

$$s_y = \frac{A}{\Phi^{-1}\left[(1+x/n)/2\right]} . \tag{5}$$

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[1] Tabulated values of $\Phi(\cdot)$ and $\Phi^{-1}(\cdot)$ can be found in most statistics textbooks.

Type B Estimation Process

As stated earlier, we will take an approach to estimating Type B uncertainties that relates to the kind of information that is normally available to technical experts. Ordinarily, technicians or engineers do not respond sensibly to questions like "in your experience, what is the uncertainty in y?" Instead, they tend to express their knowledge of such uncertainty in statements like "out of n observations on the variable y, approximately x have been found to lie within $\pm A$;" or "y lies between $\pm A$ in approximately x out of n cases;" or "y lies between $\pm A$ about p percent of the time;" etc. We see that such statements can be loosely viewed as experimental results of Bernoulli trials as discussed above.

Responses of the "x out of n" variety can form the basis for estimation of uncertainty using Eq. (4). If Bernoulli trials are systematically observed and recorded, such estimates may be regarded as Type A. If, on the other hand, Bernoulli trials are recollected as an impression based on experience, then the estimates are Type B. In both cases, it is possible to determine workable estimates of the degrees of freedom.

Type B Degrees of Freedom

In Appendix G of *The Guide to the Expression of Uncertainty in Measurement* [1], a relation is given for calculating a Type B degrees of freedom:

$$\nu_B \cong \frac{1}{2}\frac{u_B^2}{\sigma^2\left(u_B\right)} \ , \tag{6}$$

where u_B is a Type B uncertainty estimate, and $\sigma^2(u_B)$ is the variance in this estimate. Information necessary for computing this variance will be expressed in three formats:

Format 1: Approximately X% ($\pm\Delta X$%) of observed values have been found to lie within the limits $\pm A$ ($\pm\Delta A$).
Format 2: Between X% and Y% of observed values have been found to lie within the limits $\pm A$ ($\pm\Delta A$).
Format 3: Approximately x out of n values have been found to lie within the limits $\pm A$ ($\pm\Delta A$).
Format 4: Approximately X % of n values have been observed to lie within the limits $\pm$ A ($\pm$ ΔA).

Computation of the Variance in the Uncertainty

We generalize Eq. (4) to read

$$u_B = \frac{A}{\varphi(p)} \tag{7}$$

where p is the containment probability, and

$$\varphi(p) = \Phi^{-1}\left[(1+p)/2\right] \ . \tag{8}$$

The error in u_B due to errors in A and φ is obtained from Eq. (7) in the usual way

$$\begin{aligned} \varepsilon(u_B) &= \left(\frac{\partial u_B}{\partial A}\right)\varepsilon(A) + \left(\frac{\partial u_B}{\partial p}\right)\varepsilon(p) \\ &= \left(\frac{\partial u_B}{\partial A}\right)\varepsilon(A) + \left(\frac{\partial u_B}{\partial \varphi}\right)\frac{d\varphi}{dp}\varepsilon(p) \\ &= \frac{\varepsilon(A)}{\varphi} - \frac{A}{\varphi^2}\frac{d\varphi}{dp}\varepsilon(p) \ , \end{aligned} \tag{9}$$

where $\varepsilon(A)$ and $\varepsilon(p)$ are errors in A and p, respectively. Assuming statistical independence between these errors, the variance in u_B follows directly:

$$\begin{aligned} \sigma^2(u_B) &= \text{var}[\varepsilon(u_B)] \\ &= \frac{u_A^2}{\varphi^2} + \frac{A^2}{\varphi^4}\left(\frac{d\varphi}{dp}\right)^2 u_p^2 \ . \end{aligned} \tag{10}$$

Dividing Eq. (10) by the square of Eq. (7), we get

$$\frac{\sigma^2(u_B)}{u_B^2} = \frac{u_A^2}{A^2} + \frac{1}{\varphi^2}\left(\frac{d\varphi}{dp}\right)^2 u_p^2 \ . \tag{11}$$

The derivative in Eq. (11) is obtained from Eq. (8). We first establish that

$$\begin{aligned}\frac{1+p}{2} &= \Phi(\varphi) \\ &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\varphi} e^{-\zeta^2/2} d\zeta \ .\end{aligned}$$

We next take the derivative of both sides of this equation with respect to p to get

$$\frac{1}{2} = \frac{1}{\sqrt{2\pi}} e^{-\varphi^2/2} \frac{d\varphi}{dp} \ ,$$

and, finally,

$$\frac{d\varphi}{dp} = \sqrt{\frac{\pi}{2}}\, e^{\varphi^2/2} \ . \tag{12}$$

Substituting Eq. (12) in Eq. (11) yields

$$\frac{\sigma^2(u_B)}{u_B^2} = \frac{u_A^2}{A^2} + \frac{1}{\varphi^2}\frac{\pi}{2} e^{\varphi^2} u_p^2 \ . \tag{13}$$

Use of Format 1

With Format 1, a technical expert is asked to provide ± limits for both the containment limits and the containment probability. These limits are used to estimate u_A and u_p. In using Format 1, the containment probability is given by

$$p = \frac{X}{100} \ , \tag{14}$$

where X is the percentage of values of y observed within $\pm A$.

If we assume that the errors in the estimates of A and p are approximately uniformly distributed within $\pm\Delta A$ and $\pm\Delta p = \pm\Delta X\% / 100$, respectively, then we can write

$$u_A^2 = \frac{(\Delta A)^2}{3}, \text{ and } u_p^2 = \frac{(\Delta p)^2}{3} \ . \tag{15}$$

Substitution in Eq. (13) gives

$$\frac{\sigma^2(u_B)}{u_B^2} = \frac{(\Delta A)^2}{3A^2} + \frac{1}{\varphi^2}\frac{\pi}{2} e^{\varphi^2}\frac{(\Delta p)^2}{3} \ . \tag{16}$$

Use of Eq. (16) in Eq. (6) yields an estimate for the Type B degrees of freedom for Format 1:

$$\begin{aligned}\nu_B &\cong \frac{1}{2}\left(\frac{\sigma^2(u_B)}{u_B^2}\right)^{-1} \\ &\cong \frac{3\varphi^2 A^2}{2\varphi^2(\Delta A)^2 + \pi A^2 e^{\varphi^2}(\Delta p)^2} \ .\end{aligned} \tag{17}$$

Note that, if ΔA and Δp are set to zero, the Type B degrees of freedom becomes infinite.

Use of Format 2

With Format 2, the variance in A is obtained as in Format 1. The containment probability p is set at the midpoint between the lower and upper percentages divided by 100, and the variable Δp is set equal to half the difference between these percentages divided by 100. All else is the same as in Format 1.

Use of Format 3

With Format 3, the variance in the containment probability p can be obtained by taking advantage of the binomial character of p:

$$u_p^2 = \frac{p(1-p)}{n} \quad . \tag{18}$$

Substitution in Eq. (13) and using Eq. (15) for the uncertainty in A yields

$$\frac{\sigma^2(u_B)}{u_B^2} = \frac{(\Delta A)^2}{3A^2} + \frac{1}{\varphi^2}\frac{\pi}{2}e^{\varphi^2}\frac{p(1-p)}{n} \quad , \tag{19}$$

and

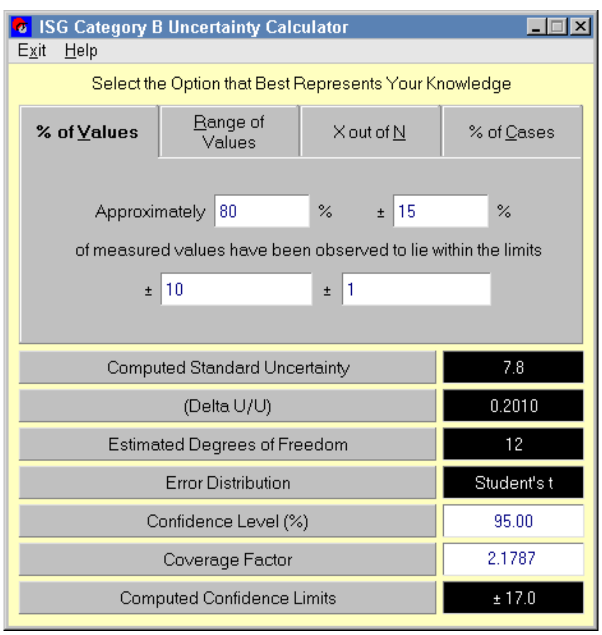
$$\begin{aligned} \nu_B &\cong \frac{1}{2}\frac{u_B^2}{\sigma^2(u_B)} \\ &\cong \frac{3\varphi^2 A^2}{2\varphi^2(\Delta A)^2 + 3\pi A^2 e^{\varphi^2} p(1-p)/n} \quad . \end{aligned} \tag{20}$$

Use of Format 4

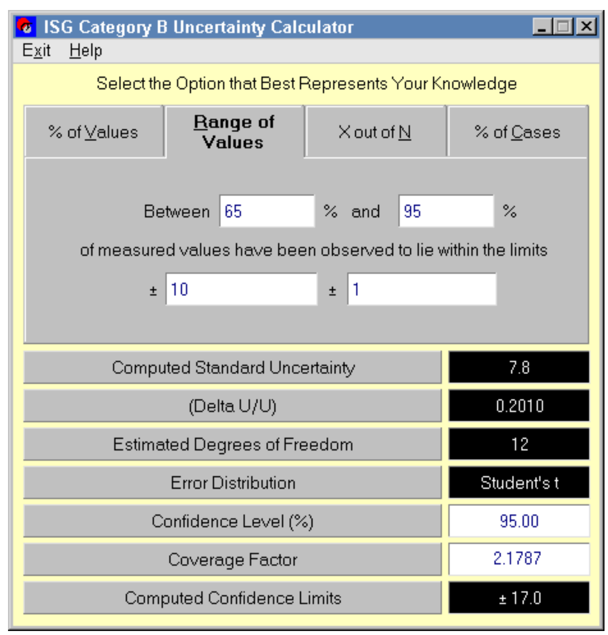
With Format 4, the variance in A is obtained as in Format 3. The containment probability p is a user-specified percentage. All else is the same as in Format 3.

Examples

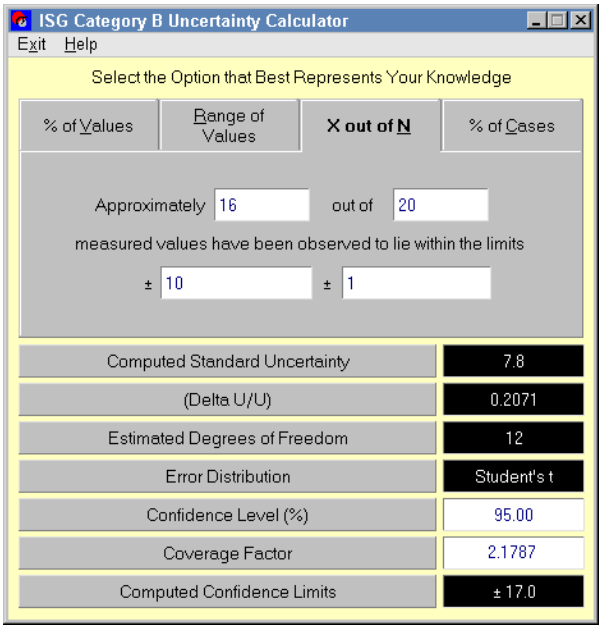
All four formats are exemplified in the figures below.



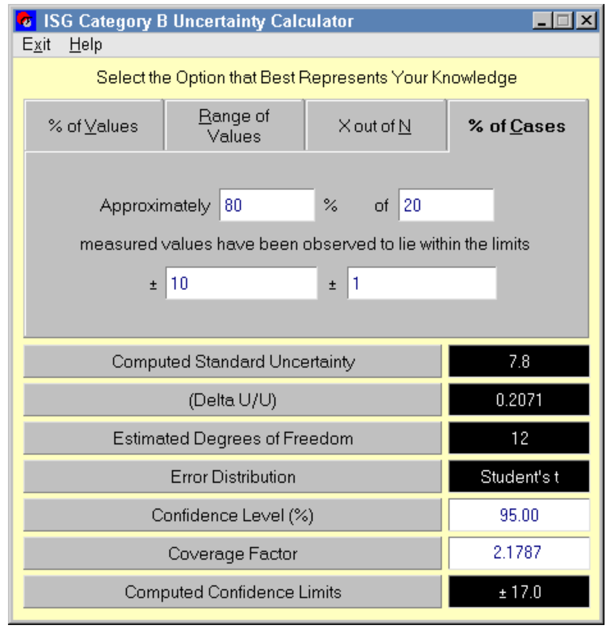
Format 1. Degrees of freedom estimate for a case where approximately 80% of observed values are recalled as being within ± 10. The approximate nature of the estimate is embodied in the error limits ± 15% and ± 1.



Format 2. Degrees of freedom estimate for a case where between 65% and 95% of observed values are recalled as being within ± 10 (±1).



Format 3. Degrees of freedom estimate for a case where approximately 16 out of 20 observed values are recalled as being within ± 10. The approximate nature of the estimate is embodied in the error limits ± 1 and in the binomial character of the estimate.



Format 4. Degrees of freedom estimate for a case where approximately 80% out of 20 observed values are recalled as being within ± 10 (± 1).

Conclusion

By obtaining values for the degrees of freedom for Type B uncertainty estimates, we place these estimates on a statistical footing. It is through the medium of the degrees of freedom statistic that the approximate nature of Type B estimates is quantified. Once this quantification has been achieved, Type B estimates can take their place alongside Type A estimates in developing confidence limits, estimating measurement decision risks and in other activities where the uncertainty estimate is taken to be a standard deviation for an underlying error distribution. This is particularly evident in combining Type A and B estimates into a total uncertainty. Given rigorous values for the degrees of freedom for both Type A and B components, the degrees of freedom for the combined total can be determined using the Welch-Satterthwaite formula. This means that the combined total may also be treated statistically.

A happy consequence of this is that we can rid ourselves of the embarrassment of arbitrary coverage factors that often bear no relationship to confidence levels or anything else of use. In addition, we no longer need to obfuscate the communication of uncertainty analysis results with the term "expanded uncertainty" to mask our inability to handle Type B estimates in a statistical way. Instead, we can return to the use of confidence limits based on considerations of uncertainty and probability. Finally, we no longer need to advise people to employ the uniform or some other simplifying distribution to estimate uncertainties in situations where these distributions are totally inappropriate.

The foregoing is not meant to imply that the problem of estimating Type B degrees of freedom has been solved and put to bed in this monograph. More research is needed in the area of extracting objective data from subjective recollections and in quantifying the lack of knowledge accompanying such data. With regard to the latter, work is required to generalize the methodology presented herein to non-normal distributions (such as may pertain to asymmetric error limits) and to the problem of combining distributions of mixed character.

References

[1] *Guide to the Expression of Uncertainty in Measurement*, 3rd Draft, ISO/TAG4/WG3, February 1990.

[2] Taylor, B. and Kuyatt, C., "Guidelines for Evaluating and Expressing the Uncertainty of NIST Measurement Results," NIST Technical Note 1297, U.S. Dept. of Commerce, January 1993.

[3] Proschan, F., "Confidence and Tolerance Intervals for the Normal Distribution," *Proc. Annual American Statistical Association*, Boston, December 1951.

[4] Youden, W., "Uncertainties in Calibration," *Proc. 1962 International Conference on Precision Electromagnetic Measurements*, August 1962.

[5] Eisenhart, C., "Realistic Evaluation of the Precision and Accuracy of Instrument Calibration Systems," J. Res. NBS, Vol. 67C, No. 2, April-June 1963.

[6] Ku, H., "Statistical Concepts in Metrology," *Handbook of Industrial Metrology*, American Society of Tool and Manufacturing Engineers, Prentice-Hall, Inc., New York, 1967.

[7] Ku, H., "Expressions of Imprecision, Systematic Error, and Uncertainty Associated with a Reported Value," *Precision Measurements and Calibration, Statistical Concepts and Procedures*, NBS Special Pub. 300, Vol. 1, Ed. by H. Ku, February 1969.

[8] *UncertaintyAnalyzer*, © 1994-2003 Integrated Sciences Group.

[9] *Type B Uncertainty Calculator*, © 1999-2003, Integrated Sciences Group, Freeware available from www.isgmax.com.

[10] Castrup, H., "Estimating Category B Degrees of Freedom," *Proc. Measurement Science Conference*, Anaheim, January 2000.